§2.1 & 2.2 THE REMAINDER THEOREM & THE FACTOR THEOREM

LONG DIVISION: Algebraic long division is very similar to traditional long division.

Example:

Divide $2x^3 - 3x^2 - 3x + 2$ by $x - 2$

$$x - 2 \overline{\big)\, 2x^3 - 3x^2 - 3x + 2}$$

What do we have to multiply x by to get $2x^3$?
Answer: $2x^2$. We put this on the top line:

$$\begin{array}{r|l} & 2x^2 \\ \hline x - 2 & 2x^3 - 3x^2 - 3x + 2 \\ & 2x^3 - 4x^2 \end{array}$$

multiply the $2x^2$ by $(x - 2)$
and write the answer here

Subtract the $2x^3 - 4x^2$ from the numbers above
& bring the next term down (the $-3x$)

$$\begin{array}{r|l} & 2x^2 \\ \hline x - 2 & 2x^3 - 3x^2 - 3x + 2 \\ & \underline{2x^3 - 4x^2} \\ & \qquad\quad x^2 - 3x \end{array}$$

Continue doing this over and over again.
What do you have to multiply x by to get x^2?

$$\begin{array}{r|l} & 2x^2 + x \\ \hline x - 2 & 2x^3 - 3x^2 - 3x + 2 \\ & \underline{2x^3 - 4x^2} \\ & \qquad\quad x^2 - 3x \\ & \qquad\quad x^2 - 2x \end{array}$$

$$\begin{array}{r|l} & 2x^2 + x \\ \hline x - 2 & 2x^3 - 3x^2 - 3x + 2 \\ & \underline{2x^3 - 4x^2} \\ & \qquad\quad x^2 - 3x \\ & \qquad\quad \underline{x^2 - 2x} \\ & \qquad\qquad\quad -x + 2 \end{array}$$

After each step, bring down the next term in the quotient. Continue until you have no terms left.

$$\begin{array}{r|l} & 2x^2 + x - 1 \\ \hline x - 2 & 2x^3 - 3x^2 - 3x + 2 \\ & \underline{2x^3 - 4x^2} \\ & \qquad\quad x^2 - 3x \\ & \qquad\quad \underline{x^2 - 2x} \\ & \qquad\qquad\quad -x + 2 \\ & \qquad\qquad\quad \underline{-x + 2} \\ & \qquad\qquad\qquad\quad 0 \end{array}$$

In this case, there is no remainder (hence the zero).

NB: If the polynomial missing power of x, add such a term by placing a zero in front of it. For example, if you are dividing $x^3 + x - 4$ by something, rewrite it as $x^3 + 0x^2 + x - 4$.

The result of the division of a polynomial function P(x) by a binomial of the form ax – b is written in **quotient form** as $\boxed{\dfrac{P(x)}{(ax - b)} = Q(x) + \dfrac{R}{(ax - b)}}$, where Q(x) is the quotient and R is the remainder.

The **corresponding statement**, that can be used to check the division, is $\boxed{P(x) = (ax - b)Q(x) + R}$.

So to check the result of a division, use: **dividend = divisor × quotient + remainder.**

SYNTHETIC DIVISION: Synthetic division is a shorthand method of polynomial division in the special case of dividing by a linear factor of the form **x + b**.

<u>Example</u>:

Use synthetic division to divide $\dfrac{2x^3 - 3x + 5}{x + 3}$.

$$x \; + \; 3 \;\Big|\; 2x^3 \; + \; 0x^2 \; - \; 3x \; + \; 5$$

Write the polynomial in descending powers of x, using zero as a placeholder for any missing terms.

$$\begin{array}{c|cccc} 3 & 2 & 0 & -3 & 5 \\ - & \downarrow & 6 & -18 & 45 \\ \hline & 2 & -6 & 15 & -40 \end{array}$$

The coefficients form the 1st row.

Subtract on the vertical.

Multiply by the "b" value on the diagonal.

Remainder

Answer: $\dfrac{2x^3 - 3x + 5}{x + 3} = 2x^2 - 6x + 15 - \dfrac{40}{x+3}$

The numbers in the third row form the coefficients of the quotient.

THE REMAINDER THEOREM

When dividing one algebraic expression by another, more often than not there will be a remainder. It is often useful to know what this remainder is and it can be calculated without going through the process of dividing as above. The rule is:

If a polynomial P(x) is divided by ax – b, the remainder is $P\left(\dfrac{b}{a}\right)$.

In the above example, $2x^3 - 3x^2 - 3x + 2$ was divided by $x - 2$.
Let $P(x) = 2x^3 - 3x^2 - 3x + 2$, $a = 1$, $b = 2$. The remainder is therefore
$P(2) = 2\times2^3 - 3\times2^2 - 3\times2 + 2 = 0$, as we saw when we divided the whole thing out.

THE FACTOR THEOREM

This states:

x - a is a factor of the polynomial P(x) if and only if P(a) = 0

Similarly, ax – b is a factor of P(x) iff $P\left(\dfrac{b}{a}\right) = 0$.

In the above worked example, $P(2) = 0$. This means that $(x - 2)$ is a factor of the equation.

INTEGRAL ZERO THEOREM

If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.

RATIONAL ZERO THEOREM

Suppose $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a zero of $P(x)$, where a and b are integers and $a \neq 0$. Then

- b is a factor of the constant term of $P(x)$
- a is a factor of the leading coefficient of $P(x)$
- $ax - b$ is a factor of $P(x)$

Sample problems:

1. Find the quotient and remainder of $x^3 + 8x^2 - 5x - 84$ divided by $x + 5$.

2. Determine if $x - 3$ is a factor of $x^3 + 5x^2 - 17x - 21$.

3. Show that $x + 2$ is a factor of $2x^3 + 3x^2 - 8x - 12$ and then factor completely.

4. Show that $x + 3$ and $x - 4$ are factors of $x^4 - 2x^3 - 13x^2 + 14x + 24$ and then factor completely.

§2.3 POLYNOMIAL EQUATIONS

KEY CONCEPTS

- The real roots of a polynomial equation P(x) = 0 correspond to the x-intercepts of the graph of the polynomial function P(x).
- The x-intercepts of the graph of a polynomial function correspond to the real roots of the related polynomial equation.
- I a polynomial equation is factorable, the roots are determined by factoring the polynomial, setting its factors equal to zero, and solving each factor.
- If a polynomial equation is not factorable, the roots can be determined from the graph using technology.

Examples:

1. Solve each of the following.

a) $x^3 - x^2 - 6x = 0$

b) $2x^3 - x^2 - 18x = -9$

c) $2x^3 - 3x^2 - 11x + 6 = 0$

2. The volume, V, in cubic centimeters, of a block of wood that a sculptor uses to carve to carve a wolf can be modeled by $V(x) = 9x^3 + 3x^2 - 120x$, where x represents the thickness of the block, in centimeters. What maximum thickness of wolf can be carved from a block of wood with volume 1332 cm^3?

§2.4 FAMILIES OF POLYNOMIAL EQUATIONS

KEY CONCEPTS

- A family of functions is a set of functions with the same characteristics.
- Polynomial functions with graphs that have the same x-intercepts belong to the same family.
- A family of polynomial functions with zeros $a_1, a_2, a_3, \ldots, a_n$ can be represented by an equation of the form $y = k(x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)$, where $k \in R$, $k \neq 0$.
- An equation for a particular member of a family of polynomial functions can be determined if a point on the graph is known.

Examples:

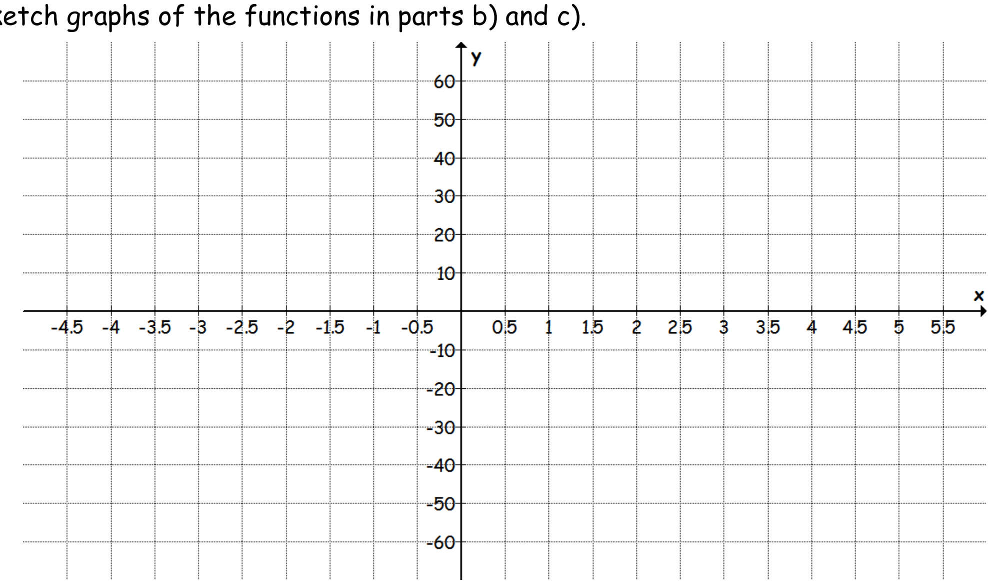
1. The zeros of a family of cubic functions are -3, 1, and 4.

a) Determine an equation for this family.

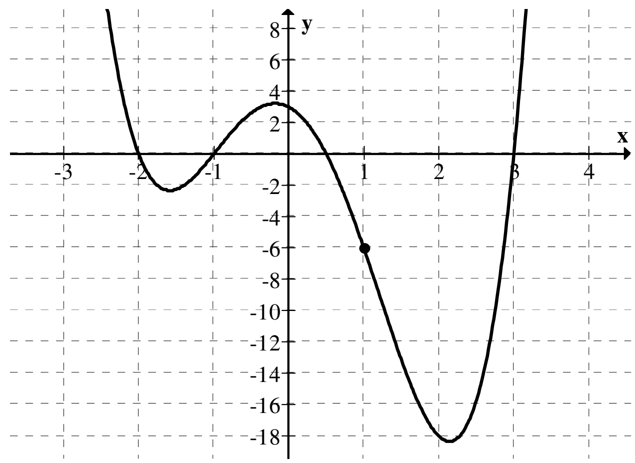
b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family whose graph has a y-intercept of -18.

d) Sketch graphs of the functions in parts b) and c).



2. Determine an equation for the quartic function represented by this graph.



3. Determine an equation, in simplified form, for the family of quartic functions with zeros 5 (order 2) and $-1 \pm 2\sqrt{2}$.

§2.6 SOLVE FACTORABLE POLYNOMIAL INEQUALITIES ALGEBRAICALLY

KEY CONCEPTS

- Factorable inequalities can be solved algebraically by
  - *Considering all cases*
  - Using intervals and then testing values in each interval
- Tables and number lines can help organize intervals to provide a visual clue to solutions.

Examples:

1. Solve the inequality $4x^3 - 7x^2 \leq 15x$ by using intervals.
Solution:
i) Compare to zero.

ii) Factor the polynomial and find the zeros. These are boundary points.

iii) Determine intervals where each factor is +/- (use a table and test values).
iv) Summarize the results.

Interval / Factor				
Result				

v) Use the results to solve the inequality.

2. The price, P, in dollars, of a stock t years after 2000 can be modeled by the function $P(t) = 0.4t^3 - 4.4t^2 + 11.2t$. When will the price of the stock be more than $36?